THEORY OF QUANTIZED HALL EFFECT AND LOCALIZATION

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1. INTRODUCTION

In a two-dimensional system in strong magnetic fields the orbital motion of electrons is completely quantized and the energy spectrum comprises discrete Landau levels. The electron localization problem in the system with such a singular density of states is not only quite interesting but also of fundamental importance. Recently, it was experimentally verified that the Hall conductivity σ_{xy} in the inversion layer made on Si-SiO_2 interface, a typical two-dimensional system, in strong magnetic fields is quantized into the integer multiple of $-e^2/h$ with $h = 2\pi\hbar$ when the Fermi level lies in the localized states existing between different Landau levels. A similar phenomenon has also been observed at a GaAs-$Al_xGa_{1-x}As$ heterojunction. Further, this fact provides a method of high precision measurement of the universal quantity e^2/h. This quantized Hall effect has a close relationship with the electron localization and/or delocalization. The purpose of this short note is to give from a theoretical point of view a review on the quantized Hall effect and the electron localization in the two-dimensional system in strong magnetic fields.

2. THEORY OF QUANTIZED HALL EFFECT

2.1 Historical Survey

Consider a two-dimensional system in a magnetic field normal to the system. When an electric field E_y is applied in the y direction, each electron moves in the x direction with a constant velocity $v_x = cE_y/H$ inversely proportional to the magnetic field H. This gives rise to the well-known expression of the Hall conductivity $\sigma_{xy} = -nec/H$, where n is the concentration of electrons in a unit area, and also the vanishing diagonal conductivity σ_{xx}. In the presence of scatterers, a frictional force acting on each electron modifies the drift motion of electrons and the Hall conductivity is no longer given by this expression. Further, the diagonal conductivity becomes nonzero.

In case of short-range scatterers the peak value of σ_{xx} has been calculated in the self-consistent Born approximation and its peak value is given by $(N+1/2)\ e^2/\pi^2\hbar$. This has been confirmed by various experiments in the silicon inversion lyaer. In 1975 Ando, Matsumoto, and Uemura calculated the Hall conductivity in several approximations. One of the important conclusion has been that the Hall conductivity is given by $\sigma_{xy} = -Ne^2/h$ when the Fermi level lies in the energy gap between the (N-1)th and Nth Landau levels. Since the total density of states of each Landau level is $1/2\pi\ell^2$, where $\ell = (c\hbar/eH)^{1/2}$ is the radius of the lowest cyclotron orbit, this leads to the interesting conclusion that the Hall conductivity when each Landau level is completely filled is not affected by the presence of impurities. This fact was confirmed experimentally by Kawaji, Igarashi, and Wakabayashi in 1975. However, this interesting phenomenon has not attracted much interest until quite recently, partly because of experimental difficulties in accurately measuring the off-diagonal conductivity. Recently, various theoretical investigations were performed to explain this interesting quantized Hall effect.

2.2 Some Exact Properties of the Hall Conductivity

Aoki and Ando have presented some exact properties of the off-diagonal conductivity σ_{xy} based on the center-migration theory of Kubo, Miyake, and Hashitsume. The results are summarized as follows: (i) As long as the Fermi level lies in the localized regime of the energy spectrum, σ_{xy} remains constant independent of n. (ii) If all the states below the Fermi level E_F are localized, we have $\sigma_{xy}=0$. These can explain the observed plateau behavior of σ_{xy} that it is independent of n in the considerable ranges of n. If we use the Kubo formula, however, nothing more can be said concerning the plateau value itself except in the case of extremely strong magnetic fields. Let us confine ourselves to the limit of strong magnetic fields, where the energy spectrum comprizes separate Landau levels and mixings between different Landau levels can be completely neglected. Aoki and Ando have shown that (iii) $\sigma_{xy} = -Ne^2/h$ when the Fermi level lies in the localized states between the Nth and (N+1)th levels. Combining the properties (ii) and (iii), we can draw an important conclusion that (v) all the states cannot be localized in the two-dimensional system in strong magnetic fields.

2.3 Laughlin's Argument

The properties (iii) and (iv) described above have been derived in the limit of strong magnetic fields where matrix elements between different Landau levels can be completely neglected. Therefore, there might be corrections due to mixings of different Landau levels. It is not an easy task to investigate corrections in the expansion with respect to $\Gamma/\hbar\omega_c$, where is broadening of a Landau level. We should remark that, within the single-site approximation, we can show that $\sigma_{xy} = -Ne^2/h$, for an arbitrary magnetic field when E_F lies in a gap region where the density of states vanishes. This strongly suggests that the plateau value of σ_{xy} is not affected by mixings between Landau levels. Laughlin has presented a plausible argument that this is actually the case. In the following a different argument originally suggested by Takemori will be presented. The argument is essentially equivalent to that of Laughlin.

We consider an L×L system and impose periodic boundary conditions in the y direction. Electrons are assumed to be free to pass through the left (x=0) and right (x=L) edges of the system. An electric field E_y is introduced by a time-dependent vector potential A_y which is independent of the coordinates x and y. A vector potential A_y independent of the coordinates can usually be eliminated by a change in the phase of the wave function (change in the gauge), i.e., by multiplying the wave function by $\exp(-ieA_y y/\hbar c)$. Because of the periodic boundary conditions in the y direction, however, the elimination is not possible except in case that A_y is the integer multiple of ΔA_y defined by $(2\pi/L)(\hbar c/e)$.

Now we increase A_y very slowly from zero. Let Δt be the time which corresponds to the change of A_y by ΔA_y. Each time A_y reaches the integer multiple of A_y, the Hamiltonian of the system becomes identical with that at t=0 (A_y=0) by the gauge transformation mentioned above. Let us now assume that the Fermi level lies in a nonzero gap in the spectrum of extended states. After the time Δt, the wave function or the density matrix of the system becomes identical with the previous one apart from a simple phase difference. This is expected since localized states do not contribute to the current and their occupancies are not influenced by the vector potential A_y at all although they lie in the vicinity of the Fermi level. Occupancy of extended states are not influenced at all because they are far from the Fermi level. The only essential change after the time Δt will be that a certain number (p) of electrons disappears through the left edge of the system and the same number of electrons enter the system through the right edge. This number of electrons, p, is considered to be independent of the time and corresponds nothing but to the Hall current. Therefore, the Hall current will be $j_x = -p(-e)/L\Delta t = -(pe/L) \times (Le/hc) (-\Delta A_y/\Delta t) = -(pe^2/h) \times E_y$, which leads to the desired quantization of the Hall conductivity into the integer multiple of $-e^2/h$.

3. MAGNETIC LOCALIZATION - NUMERICAL EXPERIMENTS

We have a nonzero σ_{xy} in the strong-field limit, which shows that all the states within a single Landau level are not localized (at least exponentially) in the present system. This indicates that the conclusion reached by Abrahams, Anderson, Licciardello, and Ramakrishnan is not applicable to the system in strong magnetic fields. This gives rise to an important theoretical question on whether there exists a mobility edge in each Landau level.

One of the most direct methods to investigate the problem of the Anderson localization is a numerical computer experiments. Aoki is the one who first made such numerical study in the two-dimensional system in strong magnetic fields. He calculated wave functions and participation ratios in several sample systems with a finite size containing randomly distributed short-range scatterers. He also calculated matrix elements of current operators. It has been suggested that states are localized near low- and high-energy edges of low-lying Landau levels. In the following, results of much more extensive numerical experiments performed by the present author will be presented.

We consider a system with a finite size L L in which scatterers are distributed randomly. We shall use periodic boundary conditions for the both x and y directions. Whether states are localized or extended is determined by a study of the Thouless number $g(L)$ which is defined as the ratio of the shifts $\langle \Delta E \rangle$ of the individual energy levels due to a change in boundary conditions to the level separation $[L^2 D(E)]^{-1}$, where $D(E)$ is the density of states per unit area. If states are localized, $g(L)$ decreases with the increase of the system size L reflecting how the wave function decays with the distance from the localization center. For extended states $g(L)$ becomes independent of L. This $g(L)$ is the scaling variable in the one-parameter scaling theory of Abrahams et al.

Here we confine ouselves to the case of relatively high concentrations of short-range scatterers $c_i = 5$, where c_i is the average number of impurities contained in the lowest cyclotron orbit. The system sizes L are chosen to be 15, 64, 100, 144, 196, 256, 324, and 400 in units of the radius of the lowest cyclotron orbit.

Figure 1 shows examples of calculate histograms of density of states for N = 0, 1, and 2 together with the results calculated in the single-site approximation. The energy is normalized by the level broadening Γ calculated in the self-consistent Born approximation and its origin is chosen at the center of each Landau level. The density of states is symmetric around the center because equal amounts of attractive and repulsive scatterers are assumed. The level width becomes slightly narrower than the dashed line obtained in the single-site approximation. The tails of the density of states become smaller and the density of states approaches that calculated in the single-site approximation with increasing level index N. This is in agreement with the result obtained previously by summing up an infinite perturbation series in an approximate way. Calculated Thouless numbers show that states are exponentially localized except in the extreme vicinity of the center of Landau levels. Figure 2 shows the inverse localization $\alpha(E)$ calculated by assuming that $g(L) = g(0) \exp[-\alpha(E)L]$. As is easily seen, the inverse localization length approaches zero smoothly as the energy approaches the center. This makes it impossible to determine the position of the critical energy E_c even if the so-called mobility edge is present. Instead, the present result strongly suggests that all the states are localized exponentially except those just at the center of the Landau level. localization length becomes zero smoothly at the center.

This is in agreement with the conclusion of Ono who calculated the dynamical conductivity $\sigma_{xx}(\omega)$ by extending a self-consistent diagramatic technique developed by Vollhardt and Wolfle to the case in strong magnetic fields. However, the energy dependence of the inverse localization length seems to be quite different from the behavior $\alpha(E) \propto \exp(-\Gamma^2/E^2)$ obtained by Ono.

4. DISCUSSION - PROBLEMS TO BE ANSWERED IN FUTURE

It is believed that states are exponentially localized in two dimensions in the absence of a magnetic field. When a weak magnetic field is applied to the system, the magnetic field tends to increase the extent of localized wave functions, which gives rise to the negative magnetoresistance widely observed in various two-dimensional systems. This is consistent with the conclusion that all the states cannot be localized in strong magnetic fields. It is believed, however, that the magnetic field does not destroy the localization as long as it is sufficiently weak. Therefore, the important question arises what the strength of the critical field is. If suh a field exists, it is determined by the level broadening Γ in case of high concentration of short-range scatterers, since Γ is, apart from the level index N, the only relevant parameter which describes interaction between different Landau levels.

It is not clear even in strong field limits that states are really extended at the center of Landau levels. It is likely that they are weakly localized (like power-law) at the center. This might be expected if we consider the case of slowly varying potential fluctuations. In this case, the electron motion becomes classical and each electron moves along an equipotential line with the velocitity proportional to the gredient of the local potential energy. It is clear that all the states are localized except those just at the center (percolation threshold) since all the equipotential lines become a closed loop. The states just at the center are considered to be localized also because the velocity of electrons vanishes at the saddle point.

The argument presented by Laughlin in explaining the quantization of the Hall conductivity into integer multiples of $-e^2/h$ seems to be quite sound. However, it may not be regarded as an exact proof since it is based on various assumptions. As far as the present author knows, no one has succeeded in proving the quantized Hall effect based on the more exact Kubo formula for the conductivity. Such a proof is highly desirable. [There is of course the possibility that the quantized Hall effect cannot obtained by the Kubo formula.]

Quite recently, Tsui, Stormer, and Gossard found a very striking result in the two-dimensional system made on the GaAs-$Al_xGa_{1-x}As$ heterojunction interface. The Hall conductivity is quantized into $-e^2/3h$ and $-2e^2/3h$ when the Fermi level lies in the lowest Landau level (N=0). It is clear that this interesting result cannot be explained by a simple one-body theory and originates from many-body electron-electron interactions. Those are all remaining important problems.

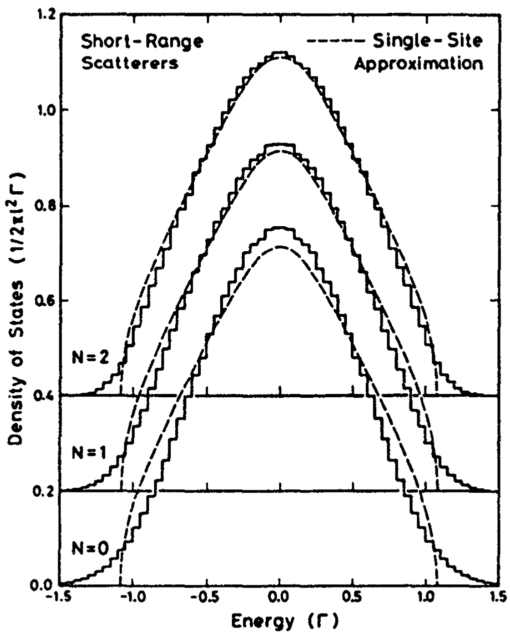

FIG. 1 Histograms of the density of states of the lowest three Landau levels N=0, 1, and 2. Short-range scatterers are assumed.

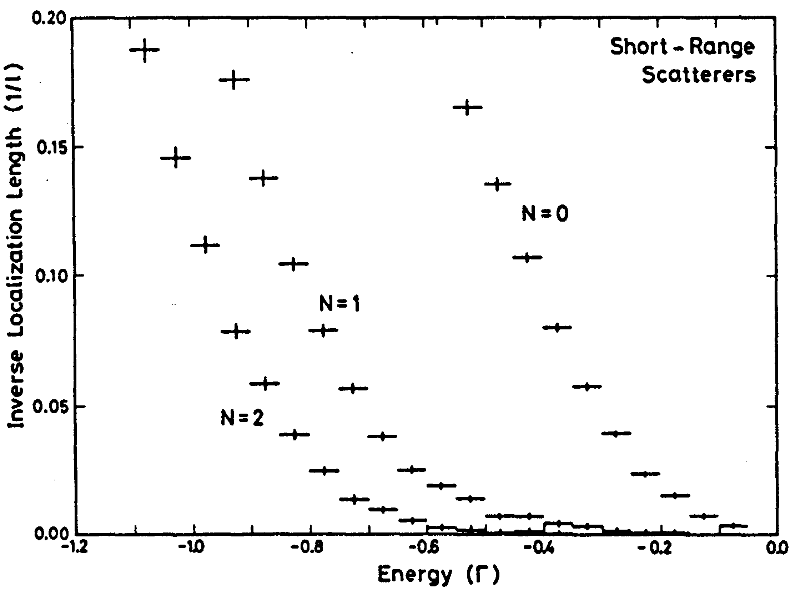

FIG. 2 Calculated inverse localization length in units of the radius of the lowest Landau level as a function of energy for the N=0, 1, and 2 Landau levels. Short-range scatterers are assumed.